Cheat Sheet: Finite Ring with Unique Zero Divisor

Given

R finite commutative ring with $1 \neq 0$, and exactly one nonzero zero divisor.

Goal

Show $|R| = 4$ and classify R.

Algorithm

1. Isolate element
Let $x \neq 0$ be the unique zero divisor.
2. Force nilpotency
Since x is a zero divisor, $\exists y \neq 0$ with $xy = 0$. Then y is a zero divisor $\Rightarrow y = x$. Hence:

$$x^2 = 0.$$

3. Control units
Let $u \in R^\times$. Then $ux \neq 0$ and

$$(ux)x = ux^2 = 0 \Rightarrow ux \text{ is zero divisor.}$$

By uniqueness: $ux = x \Rightarrow (u-1)x = 0$.
4. Structure collapse
Elements are forced into:

$$R = \{0, 1, x, 1 + x\} \Rightarrow |R| = 4.$$

5. Classification
Since $x^2 = 0$ and $x \neq 0$, R is a local ring of order 4:

$$R \cong \mathbb{Z}_4 \quad \text{or} \quad \mathbb{F}_2[X]/(X^2).$$

Distinguish by characteristic:

$$\text{char}(R) = 2 \Rightarrow \mathbb{F}_2[X]/(X^2), \quad \text{char}(R) = 4 \Rightarrow \mathbb{Z}_4.$$

Triggers

- uniqueness of zero divisor $\Rightarrow$ collapse argument - finite ring $\Rightarrow$ unit/zero divisor dichotomy - multiplication by units preserves structure